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Langevin Equation of Collective Modes of Bose–Einstein Condensates in Traps

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A quantum Langevin equation for the amplitudes of the collective modes in Bose–Einstein condensate is derived. The collective modes are coupled to a thermal reservoir of quasi-particles, whose elimination leads to the quantum Langevin equation. The dissipation rates are determined via the correlation function of the fluctuating force and are evaluated in the local-density approximation for the spectrum of quasi-particles and the Thomas–Fermi approximation for the condensate.

I take great pleasure in dedicating this paper to Gregoire Nicolis on the occasion of his sixtieth birthday.

KEY WORDS: Bose–Einstein condensates; collective modes; dissipation and fluctuation.

1. INTRODUCTION

The realization of Bose–Einstein condensates of very rarefied evaporatively cooled gases of alkali atoms in magnetic traps[(1)] offers the unique possibility to test ab initio manybody theories in the laboratory.[(2)] One very fertile field has been the experimental and theoretical investigation of collective modes of the condensates, both for zero and finite temperatures. For reviews of the experimental and theoretical work see ref. 3 and refs. 4, 5 respectively. As opposed to conventional superfluids like He-II[(6, 7)] in the new systems the collision-less regime is very naturally realized. In this regime the dominant damping mechanism for collective modes is Landau-damping, whose temperature dependence in spatially homogeneous condensates in the regime $k_B T$ large compared to the chemical potential μ has

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first been studied by Szépfalusy and Kondor.[8] Recent investigations,[9–12] though more exact, led to similar results, differing by a prefactor close to 1 for the damping rate. For condensates in traps Landau damping of low-lying modes is more difficult to calculate, and additional approximations are needed to cope with the fact that momentum is not conserved in a trap. The damping rate of collective modes in traps has been calculated in ref. 13 using the local density approximation and, in addition, a classical approximation for the correlation function whose Fourier-transform determines the cross-section of Landau-scattering. For the isotropic breathing mode in isotropic traps the Landau-damping has been calculated numerically[14] by evaluating the coupling to a great number of discrete quasi-particle modes and subsequently introducing some smoothing. The quasi-continuum coupled to the collective mode under study was displayed explicitly in this work. Theories using an extension of the approach of ref. 8 via the dielectric formalism[15, 16] and an approach via a time-dependent mean field scheme[17] have also been given.

In the present paper a very direct approach[18] to the dissipative equilibrium and nonequilibrium dynamics of collective modes in trapped Bose–Einstein condensates via quantum Langevin equations is put forward. Because of the discreteness of the mode-spectrum in traps the problem is formally similar to the quantum-optical problem of discrete modes in a laser, for which the formulation in terms of quantum Langevin equations has been very useful.[19]

In the next section the microscopic description of a trapped Bose–Einstein condensed gas is briefly set up. Then we recall the basics of the quantum Langevin equation of a boson mode. The derivation of the quantum Langevin equation of a collective mode follows. The damping rates are then evaluated in the local density and the Thomas–Fermi approximation. The last section contains a discussion of our results.

2. MICROSCOPIC EQUATIONS OF MOTION

The weakly interacting Bose-gas in a trap in standard notation is described by the Hamiltonian

$$\hat{H} = \int d^3x\, \hat{\psi}^+ \left\{ -\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{x}) - \mu + \frac{U_0}{2} \hat{\psi}^+ \hat{\psi} \right\} \hat{\psi} \tag{1}$$

The presence of a Bose–Einstein condensate means that many ($N_0 \gg 1$) particles occupy a single normalized mode of a macroscopic classical matter wave, determined as the mode of lowest energy of the classical

Hamiltonian corresponding to Eq. (1). It satisfies a classical wave equation, the Gross–Pitaevskii equation,[21] which we take in an extension defined by the so-called Popov-approximation,[22] including the interaction of the condensate with the density n' of thermal atoms, but neglecting its interaction with the pair amplitude $\langle \hat{\psi}\hat{\psi} \rangle - \psi_0^2$ of thermal particles

$$-(\hbar^2/2m)\,\nabla^2\psi_0 + (V(\mathbf{x}) + U_0(N_0\,|\psi_0(\mathbf{x})|^2 + 2n'(\mathbf{x})))\,\psi_0 = \mu\psi_0 \qquad (2)$$

For given N_0 the chemical potential μ follows by imposing the normalization condition on ψ_0.

The presence of the highly occupied condensate mode makes the decomposition of the Heisenberg field-operator

$$\hat{\psi}(\mathbf{x}, t) = (\sqrt{N_0}\,\psi_0(\mathbf{x}) + \hat{\psi}'(\mathbf{x}, t))\exp(-i\mu t/\hbar)\exp(i\phi) \qquad (3)$$

useful. $\sqrt{N_0}\exp(i\phi)$ is the complex amplitude of the classical condensate mode in equilibrium. $\hat{\psi}'(\mathbf{x}, t)$ is the field operator for the particles outside the condensate. We shall neglect fluctuations of the number of atoms in the condensate and also fluctuations of the phase of the condensate, which can be shown to occur on a time-scale much longer than the relaxation-time of the collective modes.[18]

The Hamiltonian splits up according to $\hat{H} = H_0 + \hat{H}_1 + \hat{H}_2 + \hat{H}_3 + \hat{H}_4$, with a c-number term H_0 which need not concern us here, and

$$\hat{H}_1 = \sqrt{N_0}\int d^3x((V(\mathbf{x}) - \mu + U_0(N_0\,|\psi_0|^2 + 2n'))\,\psi_0^*\hat{\psi}'$$
$$+\,(\text{hermitian conjugate}))$$

$$\hat{H}_2 = \int d^3x\left(\hat{\psi}'^+\left(-\frac{\hbar^2\nabla^2}{2m} + V(\mathbf{x}) + 2U_0 n' - \mu\right)\hat{\psi}'\right.$$
$$\left. + \frac{U_0 N_0}{2}(\psi_0^{*2}\hat{\psi}'^2 + \psi_0^2\hat{\psi}'^{+2} + 4\,|\psi_0|^2\,\hat{\psi}'^+\hat{\psi}')\right)$$

$$\hat{H}_3 = U_0\sqrt{N_0}\int d^3x(\psi_0^*(\hat{\psi}'^+\hat{\psi}' - 2n')\times\hat{\psi}' + (\text{hermitian conjugate}))$$

$$\hat{H}_4 = \frac{U_0}{2}\int d^3x(\hat{\psi}'^+\hat{\psi}'^+\hat{\psi}'\hat{\psi}' - 4n'\hat{\psi}'^+\hat{\psi}')$$

The splitting is here done in such a way that the term $\hat{H}_1$ vanishes due to Eq. (2) and the part $\hat{H}_2$ describes the linearized quantum excitations around the solution of (2). $\hat{H}_2$ is diagonalized by introducing quasi-particle operators $\hat{\alpha}_\nu$, $\hat{\alpha}_\nu^+$ by the standard Bogoliubov transformation

$$\hat{\psi}'(\mathbf{x}, t) = \sum_\nu (u_\nu(\mathbf{x})\, \hat{\alpha}_\nu(t) + v_\nu^*(\mathbf{x})\, \hat{\alpha}_\nu^+(t))$$

where u_ν, v_ν satisfy the usual Bogoliubov–De Gennes equations

$$\begin{pmatrix} -\frac{\hbar^2}{2m}\nabla^2 + U_{\text{eff}}(\mathbf{x}) - \hbar\omega_\nu & K(\mathbf{x}) \\ K^*(\mathbf{x}) & -\frac{\hbar^2}{2m}\nabla^2 + U_{\text{eff}}(\mathbf{x}) + \hbar\omega_\nu \end{pmatrix} \begin{pmatrix} u_\nu(\mathbf{x}) \\ v_\nu(\mathbf{x}) \end{pmatrix} = 0 \tag{4}$$

with the abbreviations

$$\begin{aligned} U_{\text{eff}}(\mathbf{x}) &= V(\mathbf{x}) + 2U_0(N_0\, |\psi_0(\mathbf{x})|^2 + n'(\mathbf{x})) - \mu \\ K(\mathbf{x}) &= N_0\, U_0 \psi_0(\mathbf{x})^2 \end{aligned} \tag{5}$$

Equation (4) is consistent with the ortho-normality conditions $\int d^3x(u_\nu u_\mu^* - v_\nu v_\mu^*) = \delta_{\nu\mu}$ and $\int d^3r(u_\nu^* v_\mu - u_\mu^* v_\nu) = 0$, which guarantee the Bose commutation relations of the α_ν, α_μ^+. The decomposition of $\hat{\psi}$ and $\hat{\psi}'$ together imply that $N = N_0 + \int n'(\mathbf{x})\, d^3x$ with $n' = \sum_\mu (\bar{n}_\mu(|u_\mu|^2 + |v_\mu|^2) + |v_\mu|^2)$.

Within Bogoliubov(–Popov) theory the terms $\hat{H}_3$, $\hat{H}_4$ of the total Hamiltonian are neglected and the quasi-particle operators $\hat{\alpha}_\nu$ in the Heisenberg-picture obey the Heisenberg equations of motion $\dot{\hat{\alpha}}_\nu = -i\omega_\nu \hat{\alpha}_\nu$. In this approximation the collective modes and the quasi-particles have infinite lifetime. In reality, however, the lifetime will be limited by the scattering of quasi-particles in any given mode ν with other quasi-particles from the thermal reservoir, which is described by $\hat{H}_3$ and $\hat{H}_4$. One way to describe this is the quantum Langevin equation.

3. QUANTUM LANGEVIN-EQUATION OF A HARMONIC OSCILLATOR

Let us recall here briefly the quantum Langevin equation, in Markoff approximation, of a harmonic oscillator as it is commonly used in quantum optics.[19, 20] For a detailed discussion of its derivation I refer to ref. 20. Equation (3.4.63) of that reference states the quantum Langevin equation in resonance or "rotting wave" approximation for a harmonic oscillator, described by the Bose operators $\hat{a}$, $\hat{a}^+$, in interaction with a thermal reservoir at temperature T. It takes the form

$$\dot{\hat{a}}(t) = -i\Omega \hat{a}(t) - \gamma \hat{a}(t) + \hat{\xi}(t) \tag{6}$$

where Ω is the frequency of the oscillator including a frequency shift due to the oscillator's coupling to a heat reservoir, γ is the damping rate, and $\hat{\xi}(t)$ is a Gaussian noise-operator. In Markoff approximation it has the correlation functions

$$\langle \hat{\xi}^{+}(t)\, \hat{\xi}(t') \rangle = \frac{2\gamma}{\exp(\hbar\Omega/k_B T) - 1}\, \delta(t - t') \tag{7}$$

ensuring the correct normally ordered expectation values in equilibrium, and

$$\langle [\hat{\xi}(t), \hat{\xi}^{+}(t')] \rangle = 2\gamma\delta(t - t') \tag{8}$$

ensuring the correct Bose commutation relations of $\hat{a}(t)$, $\hat{a}^{+}(t)$ for all times. The fluctuation-dissipation relation therefore permits us to infer the properties of $\hat{\xi}(t)$ if the coefficient of the dissipative term is known. Alternatively we can infer the dissipation rate γ from a microscopic expression for $\hat{\xi}(t)$ either by using (7) or, alternatively, (8).

4. QUANTUM LANGEVIN-EQUATION OF COLLECTIVE MODES

We shall here confine our attention to the dynamics of the low-lying collective modes in the collision-less regime. The interaction of the collective modes with the thermal quasi-particles is described by the Hamiltonian $\hat{H}_3 + \hat{H}_4$ not yet contained in the Bogoliubov(–Popov) approximation. Because it contains the large factor $\sqrt{N_0}$ the contribution $\hat{H}_3$ dominates over $\hat{H}_4$ and the latter can be neglected in the following. Inserting the Bogoliubov transformation in $\hat{H}_3$ and going to the interaction picture with respect to the unperturbed Bogoliubov–Popov Hamiltonian, $\hat{\tilde{H}}_3$ in interaction representation takes the form

$$\hat{\tilde{H}}_3 = \frac{\sqrt{N_0}}{2} \sum_{\kappa\nu\mu} \{ (M^{(1)}_{\kappa,\,\nu\mu} + (M^{(2)}_{\nu\mu,\,\kappa})^*)\, \hat{\alpha}^{+}_{\kappa}\, \hat{\alpha}_{\nu} \hat{\alpha}_{\mu} \exp[i(\omega_\kappa - \omega_\nu - \omega_\mu)\, t] \\ + (\text{hermitian conjugate}) \} + (\text{nonresonant terms}) \tag{9}$$

where $\bar{n}_\mu = 1/(\exp(\hbar\omega_\mu/k_B T) - 1)$ is the thermal number of quasi-particles at frequency ω_μ.

Nonresonant terms, in which the frequencies of the quasi-particles cannot add up to zero, have not been written out explicitly, because later-on we shall restrict ourselves to the resonance or rotating wave approximation in which they don't contribute. The relevant matrix elements $M^{(1)}$, $M^{(2)}$ are

$$M^{(1)}_{\kappa,\,\nu\mu} = 2U_0 \int d^3x\, \psi_0 v_\nu (u^*_\kappa u_\mu + \tfrac{1}{2} v^*_\kappa v_\mu) + (\nu \leftrightarrow \mu)$$
$$M^{(2)}_{\nu\mu,\,\kappa} = 2U_0 \int d^3x\, \psi_0 u^*_\nu (v^*_\mu v_\kappa + \tfrac{1}{2} u^*_\mu u_\kappa) + (\nu \leftrightarrow \mu) \qquad (10)$$

$M^{(1)}_{\kappa,\,\nu\mu}$ describes a scattering process in which one atom is scattered out of the condensate by the absorption of the two quasi-particles ν, μ out of and the emission of the new quasi-particle κ into the thermal bath. Likewise $M^{(2)}_{\nu\mu,\,\kappa}$ describes a scattering process where an incoming thermalquasiparticle κ is absorbed, again an atom is kicked out of the condensate, and two quasi-particles ν, μ are emitted into the thermal bath. The scattering amplitudes for both processes are linearly superposed due to the phase-coherence of the condensate on the time-scale of the relaxation process induced by the scattering process.

Taking $\hat{\hat{H}}_3$ into account the equations of motion of $\hat{\hat{\alpha}}_\nu(t)$ in the interaction picture $\hat{\alpha}_\nu(t) = \hat{\hat{\alpha}}_\nu(t) \exp(-i\omega_\nu t)$ become

$$\dot{\hat{\hat{\alpha}}}_\nu = -\frac{i}{\hbar}\frac{\sqrt{N_0}}{2} \sum_{\kappa\mu} \{ [M^{(1)}_{\nu,\,\kappa\mu} + (M^{(2)}_{\kappa\mu,\,\nu})^*]\, \hat{\hat{\alpha}}_\kappa \hat{\hat{\alpha}}_\mu \exp[i(\omega_\nu - \omega_\mu - \omega_\kappa)\, t]$$
$$+ 2[(M^{(1)}_{\kappa,\,\nu\mu})^* + M^{(2)}_{\nu\mu,\,\kappa}]\, \hat{\hat{\alpha}}^+_\mu \hat{\hat{\alpha}}_\kappa \exp[i(\omega_\nu + \omega_\mu - \omega_\kappa)\, t]\} \qquad (11)$$

If the back-action of the collective mode on the quasi-particle operators $\hat{\alpha}_\mu$, $\hat{\alpha}_\kappa$ in (11) can be ignored, the new term in this equation of motion acts like an effective random force operator. In the resonance approximation the average of this force operator vanishes. In addition it is white noise, in good approximation, if the frequencies $\omega_\kappa - \omega_\mu - \omega_\nu$ and $\omega_\kappa + \omega_\mu - \omega_\nu$ it contains form a closely spaced quasi-continuum near 0 in a neighborhood which is broad compared to the resulting damping rate γ_ν. For an explicit display of this quasi-continuum in a concrete example see ref. 14. In as much as this condition is satisfied for large condensates the Markoff-assumption made earlier is justified. All terms in the fluctuating force term not containing frequencies near frequency 0 are non-resonant and can be omitted in comparison with resonant terms.

As we recalled in the previous section, the noise term is always accompanied by a dissipative term, and, due to the Kramers–Kronig relation, also by a frequency shift. Thus the complete quantum Langevin equation in resonance approximation and in Markoff approximation must take the form of (6)

$$\dot{\hat{\alpha}}_\nu = -i(\omega_\nu + \delta_\nu)\,\hat{\alpha}_\nu - \gamma_\nu \hat{\alpha}_\nu + \hat{\xi}_\nu(t) \tag{12}$$

where $\hat{\xi}_\nu(t)$ is given by the second term in (11).

The damping rates γ_ν will be derived below, but we can also simply use (8) and represent them in the form

$$\gamma_\nu = \tfrac{1}{2}\int_{-\infty}^{+\infty} dt \langle [\hat{\xi}_\nu(t), \hat{\xi}_\nu^+(0)]\rangle \tag{13}$$

Evaluating the commutator, taking the thermal expectation value, and performing the time integral in (13) we obtain

$$\gamma_\nu = \frac{\pi N_0}{\hbar^2} \sum_{\kappa,\mu} \{ |(M^{(1)}_{\kappa,\nu\mu})^* + M^{(2)}_{\nu\mu,\kappa}|^2 (\bar{n}_\mu - \bar{n}_\kappa)\,\delta(\omega_\kappa - \omega_\mu - \omega_\nu)$$
$$+ |M^{(1)}_{\nu,\kappa\mu} + (M^{(2)}_{\kappa\mu,\nu})^*|^2 (\bar{n}_\kappa + \tfrac{1}{2})\,\delta(\omega_\kappa + \omega_\mu - \omega_\nu)\} \tag{14}$$

The first term describes Landau-damping of the mode ν by scattering a quasi-particle from mode μ to mode κ and is equivalent to a result derived in ref. 10 by the golden rule. The second term in Eq. (14) describes Beliaev damping, where the mode ν decays into two modes κ, μ. It survives even for $T \to 0$ where $\bar{n}_\kappa \to 0$ for all modes. However, for low-lying modes in traps there are only very few modes, or no modes at all, into which decay under energy conservation can occur, and this contribution to the damping is then negligible.

Let us now *derive* the dissipative term of the quantum Langevin equation. To this end we consider the equations of motion for $(d/dt)(\hat{\tilde{\alpha}}_\mu^+ \hat{\tilde{\alpha}}_\kappa)$ and $(d/dt)(\hat{\tilde{\alpha}}_\mu \hat{\tilde{\alpha}}_\kappa)$, keeping again only the resonant terms. Integrating these equations over time from $-\infty$ to t and inserting the result back into the equation of motion for $\hat{\tilde{\alpha}}_\nu$ we obtain

$$\frac{d}{dt}\hat{\tilde{\alpha}}_\nu = -\hat{\tilde{\alpha}}_\nu \frac{\sqrt{N_0}}{\hbar^2} \sum_{\mu\kappa} \left(\frac{(\bar{n}_\mu + 1/2)\,|M^{(1)}_{\nu,\kappa\mu} + (M^{(2)}_{\kappa\mu,\nu})^*|^2}{\varepsilon + i(\omega_\mu + \omega_\kappa - \omega_\nu)} + \frac{(\bar{n}_\mu - \bar{n}_\kappa)\,|(M^{(1)}_{\kappa,\nu\mu})^* + M^{(2)}_{\nu\mu,\kappa}|^2}{\varepsilon + i(\omega_\kappa - \omega_\mu - \omega_\nu)} \right) \tag{15}$$

$$-\frac{i}{\hbar}\frac{\sqrt{N_0}}{2} \sum_{\kappa\mu} \{ [M^{(1)}_{\nu,\kappa\mu} + + (M^{(2)}_{\kappa\mu,\nu})^*]\,\hat{\tilde{\alpha}}_\kappa(-\infty)\,\hat{\tilde{\alpha}}_\mu(-\infty)$$
$$\times \exp[i(\omega_\nu - \omega_\mu - \omega_\kappa)\,t] + 2[(M^{(1)}_{\kappa,\nu\mu})^* + M^{(2)}_{\nu\mu,\kappa}]\,\hat{\tilde{\alpha}}_\mu^+(-\infty)\,\hat{\tilde{\alpha}}_\kappa(-\infty)$$
$$\times \exp[i(\omega_\nu + \omega_\mu - \omega_\kappa)\,t]\} \tag{16}$$

where the limit $\varepsilon \to +0$ is implied. The second term on the right hand side is the fluctuating force term again, now more rigorously expressed in terms of the reservoir operators at the initial time at $-\infty$. Taking the limit with $(\varepsilon - i\omega)^{-1} \to \pi\delta(\omega) + iP/\omega$, where P/ω denotes the principal part under a frequency integral, we obtain the result (14) for the damping rate and also the frequency shifts δ_ν in the quantum Langevin equation. They are given by the Kramers–Kronig relation

$$\delta_\nu = -\frac{1}{\pi} P \int d\omega \, \frac{\gamma(\omega)}{\omega - \omega_\nu} \tag{17}$$

where we defined $\omega_\nu \gamma(\omega_\nu) = \gamma_\nu$.

5. DAMPING RATES

In the following we shall neglect the second term in (14), because as discussed it cannot contribute for low lying modes. Our goal in this section is the evaluation of the first term in (14) in a well defined approximation, the local density and the Thomas–Fermi approximation. The local density approximation amounts to the treatment of the quasi-continuum of the spectrum of frequencies $\omega_\kappa - \omega_\mu - \omega_\nu$ as a continuum whose density is given by the semiclassical mode-densities of the frequencies $\omega_\mu, \omega_\kappa$. Why these frequencies lie much higher than the collective mode frequency ω_ν will become clear below. The Thomas–Fermi approximation applies to large condensates[24] and amounts to neglecting the kinetic energy term in the Gross–Pitaevskii equation. The collective modes satisfy $E_\nu \ll \mu = U_0 \, |\psi_0(0)|^2$ and can be represented as[4]

$$\begin{aligned} u_\nu(\mathbf{x}) &= \left(\sqrt{\frac{U_0 n_0(\mathbf{x})}{2\hbar\omega_\nu}} + \frac{1}{2}\sqrt{\frac{\hbar\omega_\nu}{2U_0 n_0(\mathbf{x})}} \right) \chi_\nu(\mathbf{x}) \\ v_\nu(\mathbf{x}) &= \left(-\sqrt{\frac{U_0 n_0(\mathbf{x})}{2\hbar\omega_\nu}} + \frac{1}{2}\sqrt{\frac{\hbar\omega_\nu}{2U_0 n_0(\mathbf{x})}} \right) \chi_\nu(\mathbf{x}) \end{aligned} \tag{18}$$

with $\int d^3x \, |\chi_\nu(\mathbf{x})|^2 = 1$. The functions of the low-lying states $\chi_\nu(\mathbf{x})$ are known[25–29] in the hydrodynamic (long-wavelength) and Thomas–Fermi approximation (and neglecting the influence of the thermal cloud which sits mainly outside the condensate and therefore has little influence on its collective excitations). In spatially isotropic parabolic traps they have the form[25]

$$\chi_\nu(\mathbf{x}) = \frac{1}{r_{TF}^{3/2}} \, P_{\ell_\nu}^{(2n_\nu)}(x/r_{TF})(x/r_{TF})_\nu^\ell \, Y_{\ell_\nu m_\nu}(\theta, \varphi) \, \Theta(1 - x/r_{TF}) \tag{19}$$

The normalized polynomials $P_\ell^{(2n)}(x)$ are known explicitly.[26]

The high-lying quasiparticle modes can be represented as[4]

$$u_\kappa(\mathbf{x}) = \frac{E_\kappa + p_\kappa^2/2m}{\sqrt{2E_\kappa p_\kappa^2/m}}\, e^{i\mathbf{p}_\kappa\cdot\mathbf{x}/\hbar}, \qquad v_\kappa(\mathbf{x}) = -\frac{E_\kappa - p_\kappa^2/2m}{\sqrt{2E_\kappa p_\kappa^2/m}}\, e^{i\mathbf{p}_\kappa\cdot\mathbf{x}/\hbar} \tag{20}$$

with the local energies in Thomas–Fermi approximation

$$E_\kappa = E(p_\kappa, \mathbf{x}) = \sqrt{\left(\frac{p_\kappa^2}{2m} + |U_0 n_0(\mathbf{x})|\right)^2 - U_0^2 n_0^2(\mathbf{x})\,\Theta(\mu - V(\mathbf{x}))} \tag{21}$$

and $n_0(\mathbf{x}) = N_0\,|\psi_0(\mathbf{x})|^2 = (\mu/U_0)(1 - \sum_i (x_i/r_{TF}^{(i)})^2)$, and $r_{TF}^{(i)} = \sqrt{2\mu/\omega_i^2}$ are the three main Thomas–Fermi radii.

Let us consider now the Landau-damping of a low-lying phonon mode ω_ν. If the modes μ, κ involved in the scattering process were also low-lying we could use Eq. (18) and would obtain, with $E_\kappa = E_\nu + E_\mu$

$$(M_{\kappa,\nu\mu}^{(1)})^* + M_{\nu\mu,\kappa}^{(2)} = \frac{3U_0}{4\sqrt{2}} \int d^3x\, \psi_0 \chi_\kappa \chi_\mu^* \chi_\nu^* \sqrt{\frac{E_\mu E_\nu E_\kappa}{U_0^3 n_0^3(x)}} \tag{22}$$

However, in the limit of low-lying modes where $E_\nu, E_\mu, E_\kappa \ll U_0 n_0$ this matrix element becomes very small, i.e., low-lying modes cannot significantly contribute to Landau damping of other low-lying modes. Therefore the relevant modes μ, κ are in fact not low lying, local density approximation is applicable, and we can use Eq. (20) for their representation. The matrix-element for $E_\nu \ll E_\mu$, E_κ can be expanded in E_ν/μ to lowest order around $E_\kappa = E_\mu$ and becomes then

$$(M_{\kappa,\nu\mu}^{(1)})^* + M_{\nu\mu,\kappa}^{(2)} = \sqrt{\frac{E_\nu U_0}{2N_0}} \int d^3x\, \chi_\nu^*(\mathbf{x}) \exp(i(\mathbf{p}_\kappa - \mathbf{p}_\mu)\cdot\mathbf{x}))\, F(E_\mu, p_\mu) \tag{23}$$

with

$$F(E_\mu, p_\mu) = \frac{p_\mu^2}{2m} \frac{3E_\mu^2 + (p_\mu^2/2m)^2}{E_\mu(E_\mu^2 + (p_\mu^2/2m)^2} \tag{24}$$

It will be very convenient later to express the product $\chi_\nu^*(\mathbf{x})\,\chi_\nu(\mathbf{x}')$ by the associated Wigner-function W_ν via

$$\chi_\nu^*(\mathbf{x})\,\chi_\nu(\mathbf{x}') = \int \frac{d^3k}{(2\pi)^3}\, e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} W_\nu\left(\frac{1}{2}(\mathbf{x}+\mathbf{x}'), \mathbf{k}\right) \tag{25}$$

In the following we denote

$$(\mathbf{x}+\mathbf{x}')/2 \to \mathbf{x}, \qquad \mathbf{x}-\mathbf{x}' \to \mathbf{r} \tag{26}$$

The rate for Landau-damping can then be written as

$$\gamma_\nu = CE_\nu^2 \int d^3x \int \frac{d^3k}{(2\pi)^3} W_\nu(\mathbf{x},\mathbf{k}) \sum_\mu \sum_\kappa F^2(E_\mu, p_\mu)$$
$$\times \int d^3r\, e^{i/\hbar(\mathbf{p}_\kappa - \mathbf{p}_\mu - \hbar\mathbf{k})\cdot\mathbf{r}} \frac{\delta(E_\kappa - E_\mu - E_\nu)}{\sinh^2(E_\mu/2k_BT)} \tag{27}$$

with

$$C = \frac{\pi}{8\hbar}\frac{U_0}{k_BT} \tag{28}$$

The sums over the states μ and κ are only symbolic, because in the local-density approximation the discrete states have been replaced by a continuum which is normalized on the δ-function. Concretely, under the integral over $\mathbf{x}$ the sums over the energy levels of the scattering quasi-particles μ, κ are in the local density approximation replaced locally by classical phase-space averages for fixed E_μ, E_κ and final integration over E_μ and E_κ which takes automatically care of the normalization on the δ-function. Thus

$$\sum_\mu \cdots \to \sum_\mu^{(\mathbf{x})} \cdots = \int dE_\mu \int \frac{d^3p_\mu}{(2\pi\hbar)^3} \delta(E_\mu - E(p_\mu, \mathbf{x})) \cdots \tag{29}$$

In the following $\sum_\mu$ and $\sum_\kappa$ will be interpreted according to Eq. (29). The spatial integration over $\mathbf{r}$ can be done and produces a momentum-conservation factor $(2\pi\hbar^3)\,\delta^{(3)}(\mathbf{p_\kappa} - \mathbf{p_\mu} - \hbar\mathbf{k})$. Next the integrations over $\mathbf{p_\kappa}$ and E_κ contained in $\sum_\kappa$ can be performed, which just cancel the δ-functions of overall momentum and energy conservation and replace everywhere else $E_\kappa \to E_\mu + E_\nu$ and $\mathbf{p_\kappa} \to \mathbf{p_\mu} + \hbar\mathbf{k}$. Then the expression for γ_ν is reduced to

$$\gamma_\nu = CE_\nu^2 \int d^3x \int \frac{d^3k}{(2\pi)^3} W_\nu(\mathbf{x},\mathbf{k}) \int dE_\mu$$
$$\times \int \frac{d^3p_\mu}{(2\pi\hbar)^3} \delta(E_\mu - E(p_\mu, \mathbf{x})) \frac{F^2(E_\mu, p_\mu)}{\sinh^2(E_\mu/2k_BT)}$$
$$\times\, \delta(E_\mu + E_\nu - E(|\mathbf{p}_\mu + \hbar\mathbf{k}|, \mathbf{x})) \tag{30}$$

Next the integration over the *directions* of $\mathbf{p}_\mu$ relative to $\mathbf{k}$ is carried out by using up the second of the two δ-functions explicitly displayed in Eq. (30). This produces a factor 2π for the azimuthal angle, and a factor $|\partial E(p_\mu, \mathbf{x})/\partial p_\mu^2|^{-1}(2p_\mu \hbar k)^{-1}$ from the integration over $\cos\theta$ between -1 and 1, where θ is the angle between $\mathbf{k}$ and $\mathbf{p}_\mu$. Finally the integration over the absolute value p_μ is done using up the last δ-function, which picks out the $\mathbf{x}$-dependent momentum-value $p_\mu^{(0)} = \sqrt{2m}\sqrt{\sqrt{E_\mu^2 + U_0^2 n_0^2(\mathbf{x})} - U_0 n_0(\mathbf{x})}$ leaving us with the expression

$$\gamma_\nu = \frac{CE_\nu^2}{4\pi^2\hbar^3}\int d^3x \int \frac{d^3k}{(2\pi)^3}\frac{W_\nu(\mathbf{x},\mathbf{k})}{\hbar k}\int \times dE_\mu \frac{F^2(E_\mu, p_\mu^{(0)})}{4(\partial E(p_\mu^{(0)}, \mathbf{x})/\partial(p_\mu^{(0)})^2)^2 \sinh^2(E_\mu/2k_BT)} \tag{31}$$

We now have to face the difficulty to evaluate the conditional average of $(\hbar k)^{-1}$. The rigorous way to do this, which unfortunately leads to multiple integrals which are tedious to evaluate, is to invert (25) which yields

$$\int \frac{d^3k}{(2\pi)^3}\frac{W_\nu(\mathbf{x},\mathbf{k})}{\hbar k} = \int d^3r \frac{\chi_\nu^*(\mathbf{x}+\mathbf{r}/2)\,\chi_\nu(\mathbf{x}-\mathbf{r}/2)}{2\pi^2\hbar r^2} \tag{32}$$

A much simpler way consists in expressing the desired average by the local sound-velocity $\sqrt{\mu/m}\,\bar{c}_\nu(\mathbf{x})$ defined by

$$\int \frac{d^3k}{(2\pi)^3}\frac{W_\nu(\mathbf{x},\mathbf{k})}{\hbar k} = \sqrt{\frac{\mu}{m}}\frac{\bar{c}_\nu(\mathbf{x})}{E_\nu}|\chi_\nu(\mathbf{x})|^2 \tag{33}$$

and estimating the dimensionless sound-velocity $\bar{c}_\nu(\mathbf{x})$ semi-classically as $\bar{c}_\nu(\mathbf{x}) \approx \sqrt{1-(\mathbf{x}/r_{TF})^2}$ with the geometrical mean Thomas–Fermi radius $r_{TF} = (2\mu/m\bar{\omega}^2)^{1/2}$. Of course the use of the semi-classical approximation for the low lying collective mode is highly questionable and cannot be quantitatively accurate. Still we may like to use it as a rough estimate in a case where an accurate evaluation is not required or too time consuming. Below we shall check this approximation in two cases, where it cannot be expected to be particularly good.

We now introduce scaled variables $\tilde{E}_\mu = E_\mu/(N_0 U_0 \psi_0^2(\mathbf{x}))$ and $\tilde{\mathbf{x}} = \mathbf{x}/r_{TF}$, $\tilde{\mathbf{x}}' = \mathbf{x}'/r_{TF}$ with dimensionless mode-functions $\tilde{\chi}_\nu(\tilde{\mathbf{x}}) = r_{TF}^{3/2}\chi_\nu(\mathbf{x})$. Altogether, using (32), we are left with the result

$$\gamma_\nu = \frac{(a^3 n_0(0))^{1/2} E_\nu^2}{2(2\pi)^{3/2}\hbar^2\bar{\omega}} H_\nu\left(\frac{k_BT}{\mu}\right) \tag{34}$$

with the dimensionless function

$$H_\nu(z) = \int d^3\tilde{x} \int d^3\tilde{x}' \frac{\tilde{\chi}_\nu^*(\tilde{\mathbf{x}})\,\tilde{\chi}_\nu(\tilde{\mathbf{x}}')(1-(\tilde{\mathbf{x}}+\tilde{\mathbf{x}}')^2/4)}{(\tilde{\mathbf{x}}-\tilde{\mathbf{x}}')^2} \cdot \frac{1}{z} \int d\tilde{E}_\mu \left(\frac{2\tilde{E}_\mu + 1 - \sqrt{\tilde{E}_\mu^2+1}}{(\tilde{E}_\mu^2+1)\sinh((1/2z)\,\tilde{E}_\mu(1-(\tilde{\mathbf{x}}+\tilde{\mathbf{x}}')^2/4))} \right)^2 \tag{35}$$

For $z \gg 1$ the functions H_ν become linear in $z = k_B T/\mu$ and reduce to

$$H_\nu(z) \asymp 3\pi z \int d^3\tilde{x} \int d^3\tilde{x}' \frac{\tilde{\chi}_\nu^*(\tilde{\mathbf{x}})\,\tilde{\chi}_\nu(\tilde{\mathbf{x}}')}{(\tilde{\mathbf{x}}-\tilde{\mathbf{x}}')^2\,(1-(\tilde{\mathbf{x}}+\tilde{\mathbf{x}}')^2/4)} \tag{36}$$

The result for the spatially homogeneous case[9] can be recovered from Eq. (36) for $k_B T \gg \mu$ by using the scaled homogeneous condensate density $1-\tilde{x}^2 \to 1$, the phonon energy $E_\nu = \sqrt{\mu/m}\,\hbar k_\nu$, and normalized plane waves to evaluate $\int d^3\tilde{x} \int d^3\tilde{x}'\,\tilde{\chi}_\nu^*(\tilde{\mathbf{x}})\,\tilde{\chi}_\nu(\tilde{\mathbf{x}}')/(\tilde{\mathbf{x}}-\tilde{\mathbf{x}}')^2 = (2\pi^2/k_\nu)\sqrt{m\bar{\omega}^2/2\mu}$ which, together with (34), (36) yields $\gamma_\nu = (3\pi/8)\,ak_\nu(k_B T/n)$.

For isotropic traps the asymptotic result (36) becomes

$$H_{n_\nu\ell_\nu m_\nu}(z) \asymp 6\pi z \frac{(2\ell_\nu+1)(\ell_\nu-m_\nu)!}{(\ell_\nu+m_\nu)!} \times \int_0^1 d\tilde{x}\,\tilde{x}^2 P_{\ell_\nu}^{(2n_\nu)}(\tilde{x})\,\tilde{x}^{\ell_\nu} \int_0^1 d\tilde{x}'\,\tilde{x}'^2 P_{\ell_\nu}^{(2n_\nu)}(\tilde{x}')\,\tilde{x}'^{\ell_\nu} \times \int_{-1}^1 \int_{-1}^1 \int_0^{2\pi} \left(\frac{d(\cos\theta)\,d(\cos\theta')\,d\phi\,P_{\ell_\nu}^{m_\nu}(\cos\theta)\,P_{\ell_\nu}^{m_\nu}(\cos\theta') \times \exp(-im_\nu\phi)}{(2\tilde{x}\tilde{x}'(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\phi)-2)^2 - (\tilde{x}^2+\tilde{x}'^2-2)^2} \right) \tag{37}$$

where the functions $P_\ell^m(\cos\theta)$ are the associated Legendre functions appearing in the spherical harmonics.

If instead of (32) we use (33) to evaluate the conditional average of $(\hbar k)^{-1}$ we obtain in place of (37)

$$H_{n_\nu\ell_\nu m_\nu}(z) \asymp z \frac{3\sqrt{2}\,\pi^3\hbar\bar{\omega}}{E_\nu} \int_0^1 d\tilde{x} \frac{\tilde{x}^2}{\sqrt{(1-\tilde{x}^2)}} (P_{\ell_\nu}^{(2n_\nu)}(\tilde{x})\,\tilde{x}^{\ell_\nu})^2 \tag{38}$$

As was already emphasized, this result can only serve as a rough estimate for (37).

In the simplest case $n_\nu = 1$, $\ell_\nu = 0$, which is the isotropic fundamental breathing mode, we have $P_0^0(\cos(\theta)) = 1$, $P_1^{(0)}(x) = \frac{3}{2}\sqrt{7}\,(1 - \frac{5}{3}x^2)$. In this case (37) can be reduced to the numerical evaluation of a two-dimensional integral and we obtain

$$\gamma_{0,0} \asymp 26.42..\omega_0 (a^3 n_0(0))^{1/2} \frac{k_B T}{\mu} \tag{39}$$

Equation (38) yields via elementary integration $\gamma_{0,0} \asymp 27.27..\omega_0 (a^3 n_0(0))^{1/2}\, k_B T/\mu$ which agrees surprisingly well with the more rigorous result (39). Can this be considered typical? The answer is negative:

The simple result (38) lends itself to further evaluation for modes with $\ell_\nu \neq 0$. For the surface modes with $n_\nu = 0$, $\ell_\nu \neq 0$ we obtain the estimate

$$\gamma_{0,\ell_\nu} \asymp \omega_0 (a^3 n_0(0))^{1/2} \frac{k_B T}{\mu} \frac{3\pi^2}{4} \sqrt{\ell_\nu}\, \frac{\Gamma(\ell_\nu + 5/2)}{\Gamma(\ell_\nu + 2)} \tag{40}$$

In this case a numerical comparison with the more accurate expression (37) for the case $\ell_\nu = 2$ shows that the latter is about 30 percent smaller, probably giving us a realistic impression of the accuracy of the approximation for $\bar{c}_\nu(\mathbf{x})$. For larger values of ℓ_ν and n_ν the accuracy of this estimate can be expected to improve.

6. DISCUSSION AND CONCLUSION

In the present paper the many-body problem of collective modes in Bose–Einstein condensates in interaction with thermal quasi-particles was addressed by a method based on the equations of motion of the quasi-particle operators. This method leads directly to a quantum Langevin equation for the creation and annihilation operators of the collective modes, containing fluctuating force terms, a dissipation term, and a frequency shift term. These quantities are related by the fluctuation-dissipation relation and the Kramers–Kronig relation. Each part of the interaction-Hamiltonian beyond the unperturbed Bogoliubov–Popov Hamiltonian in principle gives rise to separate contributions to all three types of terms. We have here considered only the most important of these, namely the part of the interaction Hamiltonian giving rise to Landau-damping.

Dissipation can arise only from energy conserving real processes, which is manifest by the appearance of the energy conserving δ-functions in the expressions for the damping rates. This means that only resonant processes can contribute to these rates. In finite systems like the trapped condensates this causes a problem, because there the mode spectrum is discrete, the spectrum of frequency differences $\omega_\kappa - \omega_\mu - \omega_\nu$ near 0 is only a quasi-continuum, and the dissipation rates in a strict sense have to vanish. In other words, in a strict sense, what is seen as dissipation can only be a "short-time" effect; waiting for a sufficiently long time interval on the order of the inverse spacing of the quasi-continuum, revivals would have to appear. These will not be seen, however, at least in large condensates to which the local density and Thomas–Fermi approximation can be applied, because not only the energy stored in the collective mode but also the thermal energy of the system is available to bring into play a large number of modes which will lead to an irretrievable dissipation of the energy over many degrees of freedom. Therefore it is reasonable in such cases, if not required, to eliminate all recurrence effects, replacing the quasi-continuum by a true continuum, which is what the local density approximation does. Using this device we have arrived at definite results for the temperature-dependent damping rates of any collective mode, in an isotropic trap, which can be evaluated by computing numerically a multidimensional definite integral, e.g., by a Monte-Carlo routine.

The different pieces of the perturbation Hamiltonian also each give rise to frequency shifts. These are generated by virtual processes which do not require energy conservation, i.e., resonance. However the effect of the non-resonant processes is suppressed by corresponding energy-denominators and small. Here we have limited our considerations only to those processes which can also become resonant. We have here not evaluated the frequency shifts further using the local density approximation as we have done for the damping rates.

Experimental results for temperature dependent damping rates and frequency shifts have been obtained for *anisotropic* traps only,[3] and we therefore refrain from a comparison with our explicit results for isotropic traps. Detailed comparisons have been made for anisotropic traps in refs. 16 and 17 where heavier and more powerful formalisms were brought to bear together with a stronger reliance on numerical work.

The goal here has been more modest, namely to use a minimum amount of numerical work and to apply the direct and intuitive quantum Langevin approach to the fluctuations, damping rates and frequency shifts of collective modes in spatially inhomogeneous trapped Bose–Einstein condensates.

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REFERENCES

1. M. H. Anderson *et al.*, *Science* **269**:198 (1995); K. B. Davis *et al.*, *Phys. Rev. Lett.* **75**:3969 (1995); C. C. Bradley, C. A. Sackett, and R. G. Hulet, *Phys. Rev. Lett.* **78**:985 (1997).
2. Bose–Einstein Condensation in Atomic Gases, in *Proceedings of the International School of Physics "Enrico Fermi,"* M.I. Iriguscio, S. Stringari, and C. E. Wieman, eds. (IOS Press, Amsterdam, 1999).
3. W. Ketterle *et al.*, in ref. 2.
4. F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, *Rev. Mod. Phys.* **71**:463 (1999).
5. A. Griffin, in ref. 2.
6. A. Griffin, *Excitations in a Bose-Condensed Liquid* (Cambridge University Press, Cambridge, 1993).
7. P. Nozières and D. Pines, *The Theory of Quantum Liquids*, Vol. II (Addison–Wesley, Reading, Massachusetts, 1990).
8. P. Szépfalusy and I. Kondor, *Ann. Phys. (NY)* **82**:1 (1974).
9. H. Shi and A. Griffin, *Phys. Rep.* **304**:1 (1998).
10. L. P. Pitaevskii and S. Stringari, *Phys. Lett. A* **235** (1997).
11. W. Liu, *Phys. Rev. Lett.* **79**:4056 (1997).
12. S. Giorgini, *Phys. Rev. A* **57**:2949 (1998).
13. P. O. Fedichev, G. V. Shlyapnikov, and J. T. M. Walraven, *Phys. Rev. Lett.* **80**:2269 (1998); P. O. Fedichev and G. V. Shlyapnikov, *Phys. Rev. A* **58**:3146 (1998).
14. M. Guilleumas and L. P. Pitaevskii, *Phys. Rev. A* **61**:013602 (1999).
15. J. Reidl, A. Csordás, R. Graham, and P. Szépfalusy, *Phys. Rev. A* **59**:3816 (1999).
16. J. Reidl, A. Csordás, R. Graham, and P. Szépfalusy, *Phys. Rev. A* **61**:043606 (2000).
17. S. Giorgini, *Phys. Rev. A* **61**:063615 (2000).
18. R. Graham, *Phys. Rev. Lett.* **81**:5262 (1998).
19. H. Haken, "*Laser Theory*" *Encyclopedia of Physics*, Vol. XXV/2c, S. Flügge, ed. (Springer, Berlin, 1970).
20. C. W. Gardiner, *Quantum Noise* (Springer, Berlin, 1991).
21. E. P. Gross, *Nuovo Cimento* **20**:454 (1961); L. P. Pitaevskii, *Zh. Eksp. Teor. Fiz* **40**:646 (1961) [*Sov. Phys. JETP* **13**:451 (1961)].
22. A. Griffin, *Phys. Rev. B* **53**:9341 (1996).
23. N. Bogoliubov, *J. Phys. USSR* **11**:23 (1947).
24. G. Baym and C. J. Pethick, *Phys. Rev. Lett.* **76**:6 (1996).
25. S. Stringari, *Phys. Rev. Lett.* **77**:2360 (1996).
26. A. L. Fetter and D. Roksahr, *Phys. Rev.* **A57**:1191 (1998).
27. M. Fliesser, A. Csordás, P. Szépfalusy, and R. Graham, *Phys. Rev. A* **56**:R2533 (1997).
28. P. Öhberg, E. L. Surkov, I. Tittonen, M. Wilkens, and G. V. Shlyapnikov, *Phys. Rev A* **56**:R3346 (1997).
29. A. Csordaś and R. Graham, *Phys. Rev. A* **59**:1477 (1999).